Research Statement
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ASYMPTOTIC PROPERTIES OF ZETA FUNCTIONS

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1. History and known results

The starting point of our research project is the classical Brauer–Siegel theorem which is one the milestones of number theory of the past century. It reflects deep connections between algebraic, arithmetical, analytic, and (in the function field case) geometric properties of global fields. Not only is the theorem a working tool in a variety of problems concerning number fields and function fields, but the underlying ideas have been recently put into much broader context, expanding from graph-theoretic constructions [ST] to security analysis of public-key cryptosystems [HM].

The Brauer–Siegel theorem states that if K runs through the sequence of number fields normal over $\mathbb{Q}$ such that $n_K/\log|D_K| \to 0$, then $\log(h_K R_K)/\log\sqrt{|D_K|} \to 1$. Here n_K, D_K, h_K and R_K are respectively the degree, the discriminant, the class number and the regulator of the field K.

In the paper [TV2] this theorem was generalized to the case of families of almost normal global fields that have the property $n_K/\log|D_K| \to \alpha > 0$. Such families are called asymptotically good and they stand at the heart of our project. The results of M. Tsfasman and S. Vlăduţ yielded quite a few concrete arithmetic applications, like new estimates for regulators. Another approach was taken in [Z1] where we managed to prove that the statement of the classical Brauer–Siegel theorem holds in the case of asymptotically bad families of almost normal number fields.

One knows that the Brauer–Siegel theorem is equivalent to a statement about the asymptotic behaviour of the residues of Dedekind zeta functions at $s = 1$. The concept of limit zeta function permitted to extend the Brauer–Siegel theorem yet further describing the asymptotic behaviour of Dedekind zeta functions for $\mathrm{Re}\, s > \frac{1}{2}$ in families of global fields (assuming the Generalized Riemann Hypothesis (GRH) in the number field case). In [Z3] this result is proven in the number field case. An explicit version of the theorem was obtained by P. Lebacque and the author in [LZ] the proof being considerably more complicated. This result can be stated as follows: for a number field K, an integer $N \geq 10$ and any $\epsilon = \epsilon_0 + i\epsilon_1$ such that $\epsilon_0 = \mathrm{Re}\,\epsilon > 0$:

$$\sum_{q\leq N} \frac{\Phi_q \log q}{q^{\frac{1}{2}+\epsilon} - 1} + Z_K\left(\frac{1}{2}+\epsilon\right) + \frac{1}{\epsilon - \frac{1}{2}} = O\left(\frac{|\epsilon|^4 + |\epsilon|}{\epsilon_0^2}(g_K + n_K \log N)\frac{\log^2 N}{N^{\epsilon_0}}\right) + O\left(\sqrt{N}\right),$$

here Φ_q is the number of prime ideals of K of norm q, $Z_K(s)$ is the logarithmic derivative of $\zeta_K(s)$ — the Dedekind zeta function of K, $g_K = \log\sqrt{|D_K|}$. Our theorem ameliorates the explicit Brauer–Siegel and Mertens theorems from [Leb].

The above results are also related to the study of Euler–Kronecker constants which was initiated by Y. Ihara in [Ih2]. The Euler-Kronecker constant γ_K of a field K can be defined as the constant coefficient in the Taylor series expansion of $\frac{\zeta_K'(s)}{\zeta_K(s)}$. The methods from [Z3] allow (under GRH) to extend Ihara's asymptotic formula for these constants from the case of function fields to the case of number fields. More precisely, we get that $\lim\limits_{i\to\infty} \frac{\gamma_{K_i}}{g_{K_i}} = -\sum\limits_q \phi_q \frac{\log q}{q-1}$, where ϕ_q are the Tsfasman–Vlăduţ invariants of the family $\{K_i\}$ defined as $\phi_q = \lim \frac{\Phi_q(K_i)}{g_{K_i}}$.

One should also mention a recent advance in a finer analysis of asymptotic properties of values zeta and L-functions in the domain $\mathrm{Re}\, s > \frac{1}{2}$ made by Y. Ihara in [Ih3] (see also [IM] where the results are extended and improved). Studying values of abelian L-function "on average", Ihara manages to get non-trivial results where the asymptotic theory in the above sense always gives 0 as the limit (families of abelian extensions are necessarily asymptotically bad).

In the last few years there has been considerable progress in the study of the related questions in the higher dimensional situation. To begin with, let us mention a rather straightforward d-dimensional generalization of the Brauer–Siegel theorem to the case of varieties over finite fields proven in [Z2]. The main formula can be stated as $\lim\limits_{i\to\infty} \frac{\log|\varkappa(X_i)|}{b(X_i)} = \sum\limits_{f>0} \phi_{q^f} \log \frac{q^{fd}}{q^{fd}-1}$, where $\varkappa(X)$ is the residue of $\zeta_X(s)$ at $s = d$ and $b(X)$ is the sum of l-adic Betti numbers of X.

Let us say some words about the works by M. Hindry, A. Pacheco, B. Kunyavskii and M. Tsfasman [Hin], [HP], [KT] where the Brauer–Siegel type conjectures were made for families of elliptic curves over global fields. These conjectures describe the asymptotic behaviour of $|Ш_E| \cdot R_E$, where $Ш_E$ is the Shafarevich–Tate group and R_E is the covolume of the Mordell–Weil latice (the regulator). The numbers $|Ш_E|$ and R_E are important arithmetic invariants of elliptic curves which give valuable information on the structure of the group of rational points on E.

In [Z4] we managed to generalize these conjectures to the case of arbitrary families of elliptic curves over function fields by introducing the notion of an asymptotically very exact family of zeta functions. We also proved several results in the direction of these conjectures, namely, an asymptotic formula for L-functions in families of elliptic curves for $\operatorname{Re} s > 1$ and an upper bound for the central values. Actually, the framework of [Z4] is larger. We provided an abstract setting for the study of asymptotic properties of families of zeta and L-functions over finite fields not only in the connection to the Brauer–Siegel theorem but also in the aspects related to point counting on higher dimensional varieties over finite fields as well as to zero distribution results. In this way we generalized the inequalities obtained by G. Lachaud and M. Tsfasman in [LT] and a result on the distribution of zeroes of L-functions of elliptic curves over $\mathbb{F}_q(t)$ due to P. Michel [Mi]. In our studies conjectures arise which are close in spirit to those from [KS].

The characteristic zero counterparts of these results are as usual much more difficult to obtain. Apart from few particular cases (e.g. symmetric power L-functions of modular forms), families L-functions of growing analytic degree (in the sense of the Selberg class) have not yet received sufficient treatment. However, there have been numerous studies of similar problems (special values, distribution of zeroes) in the case of L-functions of modular forms and Dirichlet characters (c.f. [IS], [ILS]) and the methods employed are analytically very subtle.

Let us say some words about examples. In the function field case we dispose of a wide range of constructions of asymptotically good families of curves: those coming from supersingular points on modular curves or Drinfeld modular curves ([Ih1], [TVZ]), the explicit iterated towers proposed by Garcia and Stichtenoth [GS] and the classfield towers [Ser]. In the case of number fields only the last construction is available. It has the disadvantage of being rather inexplicit and the invariants of such infinite towers are difficult to control.

Let us finally remark that the asymptotic theory of global fields and varieties over them has particularly nice applications to coding theory and to the theory of sphere packings (c.f. [TVN]). We note that studies in the asymptotic theory of codes and packings from higher dimensional varieties are rather scarce and the field deserves further development.

2. Main goals of the project

In this project we plan develop further the asymptotic theory of zeta and L-functions, especially in the case of characteristic zero. We will pay particular attention to finding arithmetic and geometric applications of the theory as well as to studying new concrete examples. Let us describe individual problems we plan to address.

1. Asymptotic theory in characteristic zero: analytic aspects.

As was noted in the historical discussion, the asymptotic theory in characteristic zero is far less developed then its positive characteristic counterpart. It is worth saying that contrary to families of L-functions of fixed analytic degree (e.g. L-functions of Dirichlet characters, L-functions of automorphic forms on GL_2) the asymptotic theory of L-functions of growing degree has been rather understudied (in particular, very little is known about asymptotically good families of L-functions apart from the work carried out for Dedekind zeta functions of number fields).

We intend to work in the Selberg class like framework as defined in [IK, Chapter V]. That is, we impose a certain number of analytical conditions on our L-functions (analytic continuation, functional equation, Euler product, Ramanujan–Peterson conjectures) which the L-functions coming from number theory and arithmetic geometry are known to satisfy (probably conjecturally). Moreover, in some cases we will have to use the Generalized Riemann Hypothesis (GRH) since even in the classical case of Dedekind zeta functions of number fields we can not entirely get rid of it.

The notions of an asymptotically exact and that of an asymptotically good families can be carried over from the case of positive characteristic using the so called analytic conductor. As in the case of general L-functions over finite fields we will need to restrict our attention to asymptotically very exact families since the positivity of coefficients of L-functions is not guaranteed.

We plan to address the following three issues:

(1) Basic inequalities;
(2) Brauer–Siegel type results;
(3) Zero distribution problems.

The first question concerns the generalizations of the well known Odlyzhko–Serre and Drinfeld–Vlăduţ inequalities for discriminants and the number of points on curves over finite fields respectively. The case of global fields is well understood thanks to M. Tsfasman and S. Vlăduţ (c.f. [TV2]), the inequalities for varieties over finite fields were treated in [LT] and in [Z4] they were established for more general zeta and L-functions. In higher dimensions in characteristic zero only few partial results are known [Me].

The second question is about the asymptotic properties of values of L-functions when we vary the object to which our L-function is associated. Here the notion of limit L-functions becomes central. We will try to obtain both limit formulas and the analogues of these results with explicit error terms.

The third question is particularly interesting and many authors devoted their attention to its study (c.f. [ILS], [KS]). Assuming GRH, we will investigate the existence of limit distributions on the critical line when we attach to each zero the delta measure, sum them up, and let vary the object to which the L-function is associated. The study has already been started in [Z5] where the case of modular forms was treated. We have got that under GRH the sequence of measures $\Delta_f = \sum\limits_{L(f,\rho)=0} \delta_{\mathrm{Im}\,\rho} \to dx$ in the space of measures of slow growth on $\mathbb{R}$ when we let vary our primitive modular form f.

We also managed to obtain uniform distribution results for particular L-functions of growing analytic degree such as $L(\mathrm{Sym}^r f, s)$, where f is a primitive modular form and $r \to \infty$. We hope that as a result of our investigation these examples will be included in a general asymptotic theory.

As for methods, we suppose that the techniques and ideas from [Me], [TV2], [LZ], [Z3], [Z4] will be crucial for carrying out the program. In particular explicit formulas à la Weil will be important.

2. Applications of asymptotic results to the study of arithmetic properties of varieties over global fields.

In this part of our investigation we plan to apply our analytic results to obtain concrete information on arithmetic invariants of algebraic varieties over global fields.

First, assuming modularity conjectures, we are going to study in more detail families of elliptic curves and abelian varieties over number fields in the spirit of [KT]. We plan to apply our results to get information about the grows of the product of the order of the Shafarevich–Tate group and the regulator. From zero distribution results for L-functions of elliptic curves we hope to extract asymptotic bounds on the ranks of these curves in families. In particular, we hope to prove that in any asymptotically very good family $\{E_i\}$ of elliptic curves $r_{E_i}/\log N_{E_i} \to 0$, where r_{E_i} is the rank and N_{E_i} is the norm of the conductor of E_i. This was done in the function field case in [Z5] but, as usual, the number field case is technically more difficult.

It would also be interesting to know what information on varieties over number fields the basic inequalities might give. In the case of varieties over finite fields they allow to improve (asymptotically) the Weil bounds for the number of points on them. The situation in the number field case deserves further investigation.

Another important and promising direction is the study of algebraic tori both over number and function fields. For algebraic tori we have the analogues of the class number and the regulator defined in [Ono] and an analytic class number formula is known in the number field case [Sh].

In [KT] it was conjectured that for $T = T_0 \times_{\mathbb{F}_q} K_i$, where T_0 is a fixed $\mathbb{F}_q$-torus and K_i an asymptotically exact family of function fields of genus g_i, we have $\lim\limits_{i\to\infty} \frac{\log h(T)}{g_i} = \lim\limits_{i\to\infty} \frac{\log\sqrt{\mathcal{D}_T}}{g_i} - \sum\limits_{f\geq 1} \phi_{q^f} \log_q \frac{|T_0(\mathbb{F}_{q^f})|}{q^{f \dim T}}$, where $\mathcal{D}_T$ is the "quasi-discriminant" of T. Following [Z4] and using the notion of asymptotically very exact families we plan to extend this conjecture to arbitrary families of tori.

The conjectures from [Hin] and [KT] for elliptic curves turned out to be inaccessible at present and the only hope is to obtain results on average. For algebraic tori the situation is different. Indeed, in the case of norm tori the rate of growth of the class numbers is known (this is equivalent to the classical Brauer–Siegel theorem) so we hope to be able to prove the conjecture of Kunyavskii–Tsfasman using the techniques from the first part of the project. Of course, in order to reduce the question to the analytic one, we will have to employ various methods from arithmetic and algebraic geometry. In particular an analogue of the analytic class number formula is to be found in the function field case. For this purpose methods and results from [Oe] and [Hin] will undoubtfully be helpful.

3. Asymptotic properties of Selberg zeta function.

To a discrete subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ of finite covolume one can associate the so called Selberg zeta function $Z_\Gamma(s) = \prod_\gamma \prod_{m=0}^{\infty} \det(1 - e^{-l(\gamma)(s+k)})$), where γ runs through the set of primitive geodesics on $\Gamma\backslash\mathbb{H}$, $\mathbb{H}$ being the upper half plane $\{z \mid \operatorname{Im} z > 0\}$ and $l(\gamma)$ denotes the length of γ. Its properties are analogous to those of zeta functions of global fields. In particular it has a meromorphic continuation to the whole complex plane, satisfies a functional equation and an analogue of Riemann hypothesis holds for it. The methods used to study Selberg zeta functions also resemble their counterparts from the theory of Dedekind zeta functions (c.f. [He1], [He2]).

We hope to be able to use our approach to obtain asymptotic results on Selberg zeta functions, the Brauer–Siegel type theorems and the theorems on the distribution of zeroes being of particular interest. Such results may be interpreted in terms of the properties of the group Γ and of the quotient space $\Gamma\backslash\mathbb{H}$. For instance, zeroes of the Selberg zeta function are nothing but the eigenvalues of the Laplace operator acting on $\Gamma\backslash\mathbb{H}$. A uniform (in Γ) bound on the number of such eigenvalues $< X$ has recently been obtained by G. Harcos. We hope to get a more precise asymptotic information on the distribution of these eigenvalues using our approach. One should also mention some results on the analogues of the Euler–Kronecker constants for Selberg zeta functions proven in [JK]. We think that our approach will allow us to generalize them both in case of cocompact subgroups and in the case of subgroups commensurable with $\mathrm{SL}_2(\mathbb{Z})$.

4. Extension of Ihara's work on M-functions to L-functions of modular forms.

In the introduction we briefly mentioned some recent developments in the finer asymptotic theory of L-functions of Dirichlet characters made by Y. Ihara [Ih3]. We plan to generalize his approach to families of L-functions of primitive modular forms. The possibility of such extension is suggested by the considerations in [ILS]. In that paper the Peterson trace formula was used to obtain results about average properties of L-functions of modular forms whereas previously these properties were (to a certain extent) known for L-functions of Dirichlet characters. The averaging techniques applied by Ihara in the case of Dirichlet characters is quite similar to those used in the questions concerning low zeroes of L-functions, so the situation is very hopeful.

We note that such results on L-functions of modular forms would allow to look from a different angle on their behaviour in the critical strip and will possibly shed some light on important problems related to moments of L-functions.

5. Asymptotic properties of families of fields coming from modular forms.

In the final part of the project our aims will be oriented towards the search of new examples of asymptotically good families. In particular, we wish to investigate the possibility of constructing asymptotically good families of number fields in the ways different from the class field towers approach. The first non-trivial example that comes into mind is the one of extensions of $\mathbb{Q}$ obtained from modular forms.

The author together with G. Wiese studied Galois representations with values in $\mathrm{GL}_2(\mathbb{F}_q)$ associated to modular forms of varying weight and level. We managed to prove that no sequence of modular forms gives an asymptotically good family of number fields in such a way. This was accomplished by using the results on wild ramification of Galois representations from [MT].

The next step will be to treat the case of Galois representations arising from modular forms with values in $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$. The main difference here is that the image is no longer in GL_2 of a field which makes novel issues emerge here. To calculate the ramification (which is essential to decide whether a family is asymptotically good) we will need to extend to this case the theory of H. Moon and Y. Taguchi from [MT]. The class field theory over cyclotomic fields and a certain amount of work with the unit groups of cyclotomic fields will be indispensable.

3. Teaching experience

The author has the following experience in organizing seminars (contributing himself by making quite a few talks):

- 2009 — present, "Monstrous Moonshine" (together with E. Smirnov), Mathematical Department of the Higher School of Economics, Moscow.
- 2008 — 2009, "Abelian varieties" (together with C. Ritzenthaler), Luminy Mathematical Institute, Marseille.
- 2007 — 2008, "Complex multiplication" (together with C. Ritzenthaler and D. Kohel), Luminy Mathematical Institute, Marseille.
- 2006 — 2007, "Algebraic Surfaces" (together with F. Edoukou), Luminy Mathematical Institute, Marseille.
- 2005 — 2006, "Automorphic forms" (together with W. Zudilin), Moscow State University, Russia.

He has also got an experience in teaching to undergraduate students:

- 2009 — present, Calculus and Geometry courses at the Mathematical Department of the Higher School of Economics, exercise sessions.
- 2003 — 2008, Algebra, Calculus and Number Theory courses at the Independent University of Moscow, exercise sessions.
- 2005 — 2006, Tutorship in the "Math in Moscow" mathematical program for foreign students, Independent University of Moscow.

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